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Fuzzy rough bi-level multi-objective nonlinear programming problems

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Abstract Fuzzy rough bi-level multi-objective nonlinear programming problem (FRBMNPP) moved toward becoming rise normally in various real applications. In this article we develop bi-level multi-objective nonlinear programming problem (BMNPP), in which the objective functions have fuzzy nature and the constraints represented as a rough set. The fuzzy objective functions converted into deterministic ones by utilizing the α-cut methodology. Thus the FRBMNPP become a rough BMNPP which is transformed into two problems corresponding to the upper and lower approximation models. The Karush-Kuhn-Tucker (KKT) method and two models of technique of order preferences by similarity to ideal solution (TOPSIS) approach are developed to solve such problem. At last, applicability and efficiency of the two TOPSIS models and KKT method, suggested in this study, are presented through an algorithm and a numerical illustration.

1. Introduction

Bi-level decision making problem (BL-DMP) is rigorously studied and focused with considerable research interest in literature [5,6,9,17] because of its wide range of applications in numerous important fields like engineering, science, finance, management, banking, economics, agriculture and so forth [6].

BL-DMP has newly manifested in decentralized department era and has become so convoluted, especially with the elaboration of economic integration and in the era of massive data [1,7]. For that, BL-DMP mechanisms have been progressed to recognize compromises between the decision makers (DM) in a hierarchical community and activate their individual decision gradually for optimizing their objectives [1–4],[6,7,19]. Ranarahu et al. [31], proposed a method for treating a multi-objective BL-DMP. Fuzzy goal programming (FGP) algorithm was developed by Baky et al. [7], for solving fuzzy BL-DMP. Youness et al. [34], exhibited Fuzzy integer BL-DMP. Ren [30], developed a method to deal with the fully fuzzy BL-DMP by applying interval programming notions. Sakawa et al. [33], suggested interactive fuzzy programming for two-level linear fractional programming issue under fuzziness. Multi-level decision-making problems (ML-DMP) were as of late concentrated by Chen and Chen [9],. Pramanik and Roy [27], proposed FGP models for solving ML-DMP. Lachhwani [17], tackled a solution for ML-DMP based on FGP approach. An interactive approach for fractional

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ML-DMP under fuzziness displayed by Osman et al. [25]. Parametric notions of fractional fuzzy ML-DMP has been introduced by Osman et al. [23].

An essential group of algorithms utilizes the KKT conditions to represent the BL-DMP as a standard nonlinear programming problem [2,13]. KKT approach is a prevalent technique to handle programming issues with hierarchical forms. The basic concept of the KKT is that it changes each follower's problem with its optimality conditions and subjoins the obtained system to the leader's problem [2,14,15].

TOPSIS, is one of celebrated multiple criteria decision making (MCDM) manners, based on the concept that the selected alternate ought to the briefest separation from the positive ideal solution (PIS) and the most remote from the negative ideal solution (NIS) [3,4],[6]. TOPSIS exchanges the m-objectives which are conflicting and non-commensurable, into a bi-objective commensurable and most of time conflicting functions. It was first developed by Lai et al. [18], for solving a multiple attributes issue. Chen [10], extended the concept of TOPSIS to solve multi-person MCDM problems in a fuzzy domain. Large scale multi-objective non-linear programming problem via TOPSIS was exhibited by Abo-Sinna et al. [4],. A modified TOPSIS method presented by Baky and Elsayed [8], for BL-DMP with vague numbers. Baky and Abo-Sinna [6], extended the TOPSIS approach for BL-DMP.

Rough set theory (RST) can be regarded as a new mathematical tool for imperfect data analysis introduced by Pawlak [29,30], to deal with uncertainty. It has many applications found in numerous fields such as decision support, engineering, environment, medicine [12,26,27]. It communicates vagueness by utilizing a limit district of a set not by means of membership function like fuzzy set. A solution of a Rough Interval three-level Quadratic Programming Problem was presented by Saad et al. [32],. Emam et al. [11], exhibited rough ML-DMP.

Many real-world problems when mathematically modeled data some of the time can't be gathered unequivocally [1–4], [6,7,19]. This uncertainty may occur in a fuzzy or rough sense or both together [7,12,27]. Since the majority of researches on BL-DMP have been focused on the deterministic version [2,3,5]; however, in reality it is usually difficult to know precisely the crisp data due to the uncertain nature of the problems. Thus, lead us to present and architect the current FRBMNPP which is not presented before in literatures.

In this paper, we introduce a FRBMNPP. In the proposed model the objective functions coefficients have a vague nature. While the set of constrains are formulated as a rough set. Firstly, the α-cut technique was applied to obtain the α-(FRBMNPP). Secondly, to tackle the roughness of the constraints the α-(FRBMNPP) is converted into an upper approximation model (UAM) and lower approximation model (LAM). Then we present three methods to solve such problem as we first solve the UAM if the obtained solution belongs to the lower approximation set end, otherwise. We have to solve the LAM. The first method is the KKT transformation which converts the problem into one level and we solve the resultant problem. In the other two methods we made further extensions of the TOPSIS approach. Finally, the fuzzy max-min method is applied in the first model and the FGP method is utilized in the second model. An algorithm to clarify the proposed models of TOPSIS approach, illustrative example and comparison with the KKT method so also is presented.

The paper is organized as follows: Following introduction, Section 2 present some notions and preliminaries. In the next section mathematical formulation of FRBMNPP was introduced. Section 4 incorporates the KKT method for the α-(FRBMNPP). The two models of TOPSIS approach and a stepwise algorithm are explained in Section 5 and Section 6. A numerical example and comparisons are worked out in Section 7. A discussion on the uncertainty presented in Section 8. Finally, some conclusions are incorporated in Section 9.

2. Notations

Definition 1 *([16])*. $\tilde{a} = (\underline{a}, a_0, \overline{a})$ is triangular fuzzy number where $\underline{a}$ is the smallest value, a_0 is the main value, $\overline{a}$ is the highest value. The membership function $\mu_{\tilde{a}}(a;\vartheta),\ \vartheta \in [0,1]\left(0 \le \mu_{\tilde{a}}(a;\vartheta) \le \delta\right)$ where ϑ is the maximum value, $a = a_0$. Then

$$\mu_{a^{\sim}}(a;\vartheta) = \begin{cases} 0, & if\ a < \underline{a},\ or\ a > \overline{a} \\ \frac{(a-\underline{a})\vartheta}{a_0-\underline{a}}, & if\ \underline{a} \le a \le a_0 \\ \frac{(\overline{a}-a)\vartheta}{\overline{a}-a_0}, & if\ a_0 \le a \le \overline{a} \end{cases} \quad (1)$$

Definition 2. The α-level set of the fuzzy parameters $\tilde{a}$, is an ordinary set $L_\alpha(\tilde{a})$ for which the degree of its membership function exceeds the level set $\alpha \in [0,1]$, where [24,26]:

$$\begin{aligned} L_\alpha\left(\tilde{a}\right) &= \{a \in R^m | \mu_{a^{\sim}}(x) \ge \alpha\} \\ &= \left\{a \in \left[\tilde{a}_\alpha^L, \tilde{a}_\alpha^U\right] | \mu_{a^{\sim}}(x) \ge \alpha\right\} \end{aligned} \quad (2)$$

It has been shown that α-cut style is generic suitable to transact with all sorts of fuzzy mathematical including n^{th} root, exponential and taking log.

Definition 3. Let Θ be the universal set, $\mathcal{R}$ be the equivalence relation on Θ, $[\theta]_{\mathcal{R}}$ be the set of equivalence class of $\mathcal{R}$, and Ω be a non-empty subset of Θ. The upper and lower approximations of the set Ω are defined as:

$$\overline{\mathcal{R}\Omega} = \left\{\theta \in \Theta : [\theta]_{\mathcal{R}} \cap \Omega \neq \varnothing\right\}$$

$$\underline{\mathcal{R}\Omega} = \left\{\theta \in \Theta : [\theta]_{\mathcal{R}} \subseteq \Omega\right\}$$

$$\Omega = \overline{\mathcal{R}\Omega} - \underline{\mathcal{R}\Omega}$$

If $\Omega \neq$ then set Ω is called rough set [28,29].

Definition 4. The collection of all sets having the same upper and lower approximations is called a rough set, denoted by $\left(\overline{\mathcal{R}\Omega}, \underline{\mathcal{R}\Omega}\right)$ [28],.

3. Mathematical formulation

FRBMNPP can be presented as follows where the leader and the follower control the decision vectors x_1 and x_2, respectively [19,21,22]:

$[1^{st}$ **level** (**Leader**)$]$

$$\underbrace{\max}_{x_1}\tilde{F}_1(x) = \underbrace{\max}_{x_1}\tilde{f}_{1j}(x) = \sum_{k=1}^{r_1}\tilde{c}_k^{1j}\prod_{l=1}^{n}x_l^{\beta_{1jkl}}, \quad j = 1,2,\cdots,q_1$$

where x_2 solves

$[2^{nd}$ **level** (**Follower**)$]$

$$\underbrace{\max}_{x_2}\tilde{F}_2(x) = \underbrace{\max}_{x_2}\tilde{f}_{2j}(x) = \sum_{k=1}^{r_2}\tilde{c}_k^{2j}\prod_{l=1}^{n}x_l^{\beta_{2jkl}}, \quad j = 1,2,\cdots,q_2 \tag{3}$$

subject to

$x \in S(x)$

where $S(x)$ is the rough set, $\underline{S}(x) \subseteq S(x) \subseteq \overline{S}(x)$, $\underline{S}(x)$ and $\overline{S}(x)$ are the lower and upper approximation sets for $S(x)$, respectively.

$$\underline{S}(x) = \{x \in R^n : g_i(x)(\leq,=,\geq)0, \quad i = 1,2\cdots,m_1\} \neq \varnothing \tag{4}$$

$$\overline{S}(x) = \{x \in R^n : G_i(x)(\leq,=,\geq)0, \quad i = 1,2\cdots,m_2\} \neq \varnothing \tag{5}$$

Also $x_1 = (x_{11}, x_{12}, \cdots, x_{1n_1}) \in R^{n_1}$, $x_2 = (x_{21}, x_{22}, \cdots, x_{2n_2}) \in R^{n_2}$, $x = (x_1, x_2) \in R^n$, $n = n_1 + n_2$ and x_l denotes any decision variable of the two decision vectors x_1 and x_2. $\tilde{f}_{ij}(x), i = 1,2, \ j = 1,2,\cdots,q_i$ is a nonlinear polynomial function with fuzzy coefficient, $\tilde{c} \in \tilde{a}$; $\beta_{ijk} \in R$. $g(x)$ and $G(x)$ are assumed to be at least second differentiable and continous functions.

Applying the α-cut technique, we obtain the deterministic form of the objective functions then we transform the α-(FRBMNPP) into an UAM and LAM as [16,22,35]:

UAM :

$$\underbrace{\max}_{x_1}\tilde{F}_1(x)=\underbrace{\max}_{x_1}\tilde{f}_{1j}(x)=\sum_{k=1}^{r_1}\left(\tilde{c}_k^{1j}\right)_\alpha^U\prod_{l=1}^{n}x_l^{\beta_{1jkl}}, \quad j=1,2,\cdots,q_1$$

where $x_2$solves

$$\underbrace{\max}_{x_2}\tilde{F}_2(x)=\underbrace{\max}_{x_2}\tilde{f}_{2j}(x)=\sum_{k=1}^{r_2}\left(\tilde{c}_k^{2j}\right)_\alpha^U\prod_{l=1}^{n}x_l^{\beta_{2jkl}}, j=1,2,\cdots,q_2$$

subject to

$x \in \overline{S}(x)$

(6)

Definition 4. For any feasible $x_1(x_1 \in \overline{S}(x))$ given by the leader if $x_2(x_2 \in \overline{S}(x))$ is the α-Pareto optimal solution of the BL-DMP, then (x_1, x_2) is an α-feasible solution of the α-(FRBMNPP) for the UAM.

Definition 5. A point x^* is an α-Pareto optimal solution of the α-(FRBMNPP) for the UAM if there exist no other α-feasible solution $x \in \overline{S}(x)$ exist, such that $\left(\tilde{f}_{1j}(x^*)\right)_\alpha^U \leq \left(\tilde{f}_{1j}(x)\right)_\alpha^U$ for at least one objective function $\left(\tilde{f}_{1j}(x)\right)_\alpha^U$.

Definition 6. x^* is called an α-surly Pareto optimal solution if and only if $x^*$is the α-Pareto optimal solution of the α-(FRBMNPP) for the UAM and $x^* \in \underline{S}(x)$. Otherwise this solution is called an α-possibly Pareto optimal solution.

LAM :

$$\underbrace{\max}_{x_1}\tilde{F}_1(x)=\underbrace{\max}_{x_1}\tilde{f}_{1j}(x)=\sum_{k=1}^{r_1}\left(\tilde{c}_k^{1j}\right)_\alpha^U\prod_{l=1}^{n}x_l^{\beta_{1jkl}}, \quad j=1,2,\cdots,q_1$$

where x_2 solves

$$\underbrace{\max}_{x_2}\tilde{F}_2(x)=\underbrace{\max}_{x_2}\tilde{f}_{2j}(x)=\sum_{k=1}^{r_2}\left(\tilde{c}_k^{2j}\right)_\alpha^U\prod_{l=1}^{n}x_l^{\beta_{2jkl}}, \quad j=1,2,\cdots,q_2$$

subject to

$x \in \underline{S}(\boldsymbol{x})$

(7)

4. KKT method for FRBMNPP

KKT method convert the α-(FRBMNPP) of the UAM and LAM into an α-multi-objective nonlinear programming problem α-(MNPP). By replacing the follower problem by its KKT optimality conditions and appends the obtained system to the leader's problem [2,14]. So, the KKT conditions of the follower problem for the UAM are:

$$\begin{aligned}
&\nabla_{x_2}\sum_{j=1}^{q_2}w_{2j}\left(\tilde{f}_{2j}(x)\right)_\alpha^U - \nabla_{x_2}\sum_{i=1}^{m_2}\mu_i G_i(x) = 0\\
&G_i(x) \leq 0, \quad i = 1,2\cdots,m_2\\
&\mu_i G_i(x) = 0, \quad i = 1,2\cdots,m_2\\
&\mu_i \geq 0, \quad i = 1,2\cdots,m_2\\
&w_2 \in [0,1], \sum_{j=1}^{q_2}w_{2j} = 1
\end{aligned} \tag{8}$$

where μ_i is a Lagrange multiplier which is associated with the inequality constraint $G_i(x)$, see [12,14]. Then, the α-(MNPP) of the UAM can be written as:

$$\underbrace{\max}_{x_1}\sum_{j=1}^{q_1}w_{1j}\left(\tilde{f}_{1j}(x)\right)_\alpha^U$$

subject to

$$\begin{aligned}
&\nabla_{x_2}\sum_{j=1}^{q_2}w_{2j}\left(\tilde{f}_{2j}(x)\right)_\alpha^U - \nabla_{x_2}\sum_{i=1}^{m_2}\mu_i G_i(x) = 0\\
&G_i(x) \leq 0, \quad i = 1,2\cdots,m_2,\\
&\mu_i G_i(x) = 0, \quad i = 1,2\cdots,m_2,\\
&\mu_i \geq 0, \quad i = 1,2\cdots,m_2,\\
&w_1, w_2 \in [0,1], \quad \sum_{j=1}^{q_1}w_{1j} = 1, \quad \sum_{j=1}^{q_2}w_{2j} = 1
\end{aligned} \tag{9}$$

5. TOPSIS for FRBMNPP

In this section, we exhibit two models for solving FRBMNPP based on the concept of TOPSIS [36],. By obtaining the α-(FRBMNPP) then we transform the problem into an UAM and LAM. Firstly, we solve the UAM by two models of TOPSIS approach if the solution is α-surely Pareto optimal we stop otherwise we solve the LAM.

5.1. The TOPSIS methodology for the leader problem of UAM

The TOPSIS approach [36], is utilized to solve the following leader problem:

$$\underbrace{\max}_{x_1} \left(\tilde{f}_{1j}(x)\right)_\alpha^U = \sum_{k=1}^{r_1} \left(\tilde{c}_k^{1j}\right)_\alpha^U \prod_{l=1}^{n} x_l^{\beta_{1jkl}}, \quad j = 1, 2, \cdots, q_1$$
$$\text{subject to}$$
$$x \in \overline{S}(x) \tag{10}$$

Architecting the bi-objective distance functions of Lai et al. [18], for the leader problem:

$$\min \ d_P^{PIS^F}(x) = \left\{ \sum_{j=1}^{q_1} \gamma_{1j}^p \left[\frac{f_{1j}^* - \sum_{k=1}^{r_1} \left(\tilde{c}_k^{1j}\right)_\alpha^U \prod_{l=1}^{n} x_l^{\beta_{1jkl}}}{f_{1j}^* - f_{1j}^-} \right]^p \right\}^{\frac{1}{p}}$$
$$\max \ d_P^{NIS^F}(x) = \left\{ \sum_{j=1}^{q_1} \gamma_{1j}^p \left[\frac{\sum_{k=1}^{r_1} \left(\tilde{c}_k^{1j}\right)_\alpha^U \prod_{l=1}^{n} x_l^{\beta_{1jkl}} - f_{1j}^-}{f_{1j}^* - f_{1j}^-} \right]^p \right\}^{\frac{1}{p}} \tag{11}$$
$$\text{subject to}$$
$$x \in \overline{S}(x)$$

where $f_{1j}^* = \max_{x \in \overline{S}(x)} \left(\tilde{f}_{1j}(x)\right)_\alpha^U$, is the individual PIS, $f_{1j}^- = \min_{x \in \overline{S}(x)} \left(\tilde{f}_{1j}(x)\right)_\alpha^U$, is the individual NIS and γ_{1j}, is the weights of the objectives. Postulate that the membership functions $\mu_F^{PIS}(x)$ and $\mu_F^{NIS}(x)$ of model (11) are linear between $\left(d_p^F\right)^* = \left(\left(d_P^{PIS^F}\right)^*, \left(d_P^{NIS^F}\right)^*\right)$ and $\left(d_p^F\right)^- = \left(\left(d_P^{PIS^F}\right)^-, \left(d_P^{NIS^F}\right)^-\right)$ which are:

$$\left(d_P^{PIS^F}\right)^* = \min_{x \in \overline{S}(x)} d_P^{PIS^F}(x) \ \& \ \left(d_P^{PIS^F}\right)^- = \max_{x \in \overline{S}(x)} d_P^{PIS^F}(x) \tag{12}$$

$$\left(d_P^{NIS^k}\right)^* = \max_{x \in \overline{S}(x)} d_P^{NIS^k}(x) \ \& \ \left(d_P^{NIS^F}\right)^- = \min_{x \in \overline{S}(x)} d_P^{NIS^F}(x) \tag{13}$$

Thus $\mu_F^{PIS}(x) \equiv \mu_{d_P^{PIS^F}}(x)$ and $\mu_F^{NIS}(x) \equiv \mu_{d_P^{NIS^F}}(x)$ can be obtained as [3,4,6]:

$$\mu_F^{PIS}(x) = \begin{cases} 1 & \text{if } d_P^{PIS^F}(x) < \left(d_P^{PIS^F}\right)^* \\ \dfrac{\left(d_P^{PIS^F}\right)^- - d_P^{PIS^F}(x)}{\left(d_P^{PIS^F}\right)^- - \left(d_P^{PIS^F}\right)^*} & \text{if } \left(d_P^{PIS^F}\right)^* \le d_P^{PIS^F}(x) \le \left(d_P^{PIS^F}\right)^- \\ 0 & \text{if } \left(d_P^{PIS^F}\right)^- < d_P^{PIS^F}(x) \end{cases} \tag{14}$$

$$\mu_F^{NIS}(x) = \begin{cases} 1 & \text{if } d_P^{NIS^F}(x) > \left(d_P^{NIS^F}\right)^* \\ \dfrac{d_P^{NIS^F}(x) - \left(d_P^{NIS^F}\right)^-}{\left(d_P^{NIS^F}\right)^* - \left(d_P^{NIS^F}\right)^-} & \text{if } \left(d_P^{NIS^F}\right)^- \le d_P^{NIS^F}(x) \le \left(d_P^{NIS^F}\right)^* \\ 0 & \text{if } d_P^{NIS^F}(x) < \left(d_P^{NIS^F}\right)^- \end{cases} \tag{15}$$

In the first model, we apply the fuzzy max-min decision model [3,6,36]. Thus model (11) is equivalent to the following Tchebycheff model:

Model (I) :

$$\max \lambda$$
$$\text{subject to}$$
$$(1-\lambda)\left(d_P^{PIS^F}\right)^- - \left\{ \sum_{j=1}^{q_1} \gamma_{1j}^p \left[\frac{f_{1j}^* - \sum_{k=1}^{r_1} \left(\tilde{c}_k^{1j}\right)_\alpha^U \prod_{l=1}^{n} x_l^{\beta_{1jkl}}}{f_{1j}^* - f_{1j}^-} \right]^p \right\}^{\frac{1}{p}} \ge -\lambda\left(d_P^{PIS^F}\right)^*$$
$$\left\{ \sum_{j=1}^{q_1} \gamma_{1j}^p \left[\frac{\sum_{k=1}^{r_1} \left(\tilde{c}_k^{1j}\right)_\alpha^U \prod_{l=1}^{n} x_l^{\beta_{1jkl}} - f_{1j}^-}{f_{1j}^* - f_{1j}^-} \right]^p \right\}^{\frac{1}{p}} + (\lambda - 1)\left(d_P^{NIS^F}\right)^- \ge \lambda\left(d_P^{NIS^F}\right)^* \tag{16}$$
$$x \in \overline{S}(x)$$

The second introduced model is based on FGP method, [7,20] to solve the conflicting bi-objective distance functions of the leader problem as follows:

Model (II) :

$$\min Z = D_F^{PIS^+} + D_F^{NIS^-}$$
$$\text{subject to}$$
$$\frac{\left(d_P^{PIS^F}\right)^- - \left\{ \sum_{j=1}^{q_1} \gamma_{1j}^p \left[\frac{f_{1j}^* - \sum_{k=1}^{r_1} \left(\tilde{c}_k^{1j}\right)_\alpha^U \prod_{l=1}^{n} x_l^{\beta_{1jkl}}}{f_{1j}^* - f_{1j}^-} \right]^p \right\}^{\frac{1}{p}}}{\left(d_P^{PIS^F}\right)^- - \left(d_P^{PIS^F}\right)^*} + D_F^{PIS^-} - D_F^{PIS^+} = 1 \tag{17}$$
$$\frac{\left\{ \sum_{j=1}^{q_1} \gamma_{1j}^p \left[\frac{\sum_{k=1}^{r_1} \left(\tilde{c}_k^{1j}\right)_\alpha^U \prod_{l=1}^{n} x_l^{\beta_{1jkl}} - f_{1j}^-}{f_{1j}^* - f_{1j}^-} \right]^p \right\}^{\frac{1}{p}} - \left(d_P^{NIS^F}\right)^-}{\left(d_P^{NIS^F}\right)^* - \left(d_P^{NIS^F}\right)^-} + D_F^{NIS^-} - D_F^{NIS^+} = 1$$
$$x \in \overline{S}(x)$$

Since, the essential idea of the BL-DMP is that the leader sets his decisions and then asks the follower for its optima [3],[5–7]. In this way, to stay away from vast computational efforts because of, models exhibited in [6,7]. The reason behind of setting the decision variables of the leader as restricting constraints to show the efficiency of the proposed fuzzy max-min method and FGP method i.e. the UAM for the α-(FRBMNPP) has an α-feasible solution even if we don't provide any leeway for the leader decision variables.

5.2. The TOPSIS models for FRBMNPP

To obtain a compromise solution to FRBMNPP using the two proposed models of TOPSIS approach, we modify the distance family, $d_P^{PIS^B}$ and $d_P^{NIS^B}$, as follows:

$$d_P^{PIS^B}(x) = \left\{ \sum_{i=1}^{2} \left(\sum_{j=1}^{q_i} \gamma_{ij}^p \left[\frac{f_{ij}^* - \sum_{k=1}^{r_1} \left(\tilde{c}_k^{ij}\right)_\alpha^U \prod_{l=1}^n x_l^{\beta_{ijkl}}}{f_{ij}^* - f_{ij}^-} \right]^p \right) \right\}^{\frac{1}{p}} \tag{18}$$

$$d_P^{NIS^B}(x) = \left\{ \sum_{i=1}^{2} \left(\sum_{j=1}^{q_i} \gamma_{ij}^p \left[\frac{\sum_{k=1}^{r_1} \left(\tilde{c}_k^{ij}\right)_\alpha^U \prod_{l=1}^n x_l^{\beta_{ijkl}} - f_{ij}^-}{f_{ij}^* - f_{ij}^-} \right]^p \right) \right\}^{\frac{1}{p}} \tag{19}$$

where $\gamma_{ij}, i = 1,2, j = 1,2,\cdots,q_i$ are the relative importance of the objectives. And also, $f_{ij}^* = x \in \overline{S}(\boldsymbol{x}) max\left(\tilde{f}_{ij}(\boldsymbol{x})\right)_\alpha^U$, $f_{ij}^- = x \in \overline{S}(\boldsymbol{x}) min\left(\tilde{f}_{ij}(\boldsymbol{x})\right)_\alpha^U, i = 1,2, j = 1,2,\cdots,q_i,$ respectively. Then the α-(FRBMNPP) model (6) transferred into the distance functions [6,34]:

$$\min d_P^{PIS^B}(x) = \left\{ \sum_{i=1}^{2} \left(\sum_{j=1}^{q_i} \gamma_{ij}^p \left[\frac{f_{ij}^* - \sum_{k=1}^{r_1} \left(\tilde{c}_k^{ij}\right)_\alpha^U \prod_{l=1}^n x_l^{\beta_{ijkl}}}{f_{ij}^* - f_{ij}^-} \right]^p \right) \right\}^{\frac{1}{p}}$$

$$\max d_P^{NIS^B}(x) = \left\{ \sum_{i=1}^{2} \left(\sum_{j=1}^{q_i} \gamma_{ij}^p \left[\frac{\sum_{k=1}^{r_1} \left(\tilde{c}_k^{ij}\right)_\alpha^U \prod_{l=1}^n x_l^{\beta_{ijkl}} - f_{ij}^-}{f_{ij}^* - f_{ij}^-} \right]^p \right) \right\}^{\frac{1}{p}} \tag{20}$$

subject to

$x \in \overline{S}(x)$

Thus we construct the membership functions $\mu_{d_P^{PIS^B}}(x)$ and $\mu_{d_P^{NIS^B}}(x)$ [3,6,18] as:

$$\mu_{d_P^{PIS^B}}(x) = \begin{cases} 1 & \text{if } d_P^{PIS^B}(x) < \left(d_P^{PIS^B}\right)^* \\ \dfrac{\left(d_P^{PIS^B}\right)^- - d_P^{PIS^B}(x)}{\left(d_P^{PIS^B}\right)^- - \left(d_P^{PIS^B}\right)^*} & \text{if } \left(d_P^{PIS^B}\right)^* \le d_P^{PIS^B}(x) \le \left(d_P^{PIS^B}\right)^- \\ 0 & \text{if } \left(d_P^{PIS^B}\right)^- < d_P^{PIS^B}(x) \end{cases} \tag{21}$$

$$\mu_{d_P^{NIS^B}}(x) = \begin{cases} 1 & \text{if } d_P^{NIS^B}(x) > \left(d_P^{NIS^B}\right)^* \\ \dfrac{d_P^{NIS^B}(x) - \left(d_P^{NIS^B}\right)^-}{\left(d_P^{NIS^B}\right)^* - \left(d_P^{NIS^B}\right)^-} & \text{if } \left(d_P^{NIS^B}\right)^- \le d_P^{NIS^B}(x) \le \left(d_P^{NIS^B}\right)^* \\ 0 & \text{if } d_P^{NIS^B}(x) < \left(d_P^{NIS^B}\right)^- \end{cases} \tag{22}$$

So, we can formulate the final fuzzy max-min model (I) and FGP model (II), respectively as follows:

Model (I) :

$\max \delta$

subjectto

$$(1-\delta)\left(d_P^{PIS^B}\right)^- - \left\{ \sum_{i=1}^{2} \left(\sum_{j=1}^{q_i} \gamma_{ij}^p \left[\frac{f_{ij}^* - \sum_{k=1}^{r_1} \left(\tilde{c}_k^{ij}\right)_\alpha^U \prod_{l=1}^n x_l^{\beta_{ijkl}}}{f_{ij}^* - f_{ij}^-} \right]^p \right) \right\}^{\frac{1}{p}} \ge -\delta\left(d_P^{PIS^B}\right)^*$$

$$\left\{ \sum_{i=1}^{2} \left(\sum_{j=1}^{q_i} \gamma_{ij}^p \left[\frac{\sum_{k=1}^{r_1} \left(\tilde{c}_k^{ij}\right)_\alpha^U \prod_{l=1}^n x_l^{\beta_{ijkl}} - f_{ij}^-}{f_{ij}^* - f_{ij}^-} \right]^p \right) \right\}^{\frac{1}{p}} + (\delta-1)\left(d_P^{NIS^B}\right)^- \ge \delta\left(d_P^{NIS^B}\right)^* \tag{23}$$

$x \in \overline{S}(x)$

$\boldsymbol{x}_1 = \boldsymbol{x}_1^*$

The second proposed model is based on FGP approach, follows as:

Model (II) :

$\min Z = D_B^{PIS^+} + D_B^{NIS^-}$

subjectto

$$\frac{\left(d_P^{PIS^B}\right)^- - \left\{ \sum_{i=1}^{2} \left(\sum_{j=1}^{q_i} \gamma_{ij}^p \left[\frac{f_{ij}^* - \sum_{k=1}^{r_1} \left(\tilde{c}_k^{ij}\right)_\alpha^U \prod_{l=1}^n x_l^{\beta_{ijkl}}}{f_{ij}^* - f_{ij}^-} \right]^p \right) \right\}^{\frac{1}{p}}}{\left(d_P^{PIS^B}\right)^- - \left(d_P^{PIS^B}\right)^*} + D_B^{PIS^-} - D_B^{PIS^+} = 1$$

$$\frac{\left\{ \sum_{i=1}^{2} \left(\sum_{j=1}^{q_i} \gamma_{ij}^p \left[\frac{\sum_{k=1}^{r_1} \left(\tilde{c}_k^{ij}\right)_\alpha^U \prod_{l=1}^n x_l^{\beta_{ijkl}} - f_{ij}^-}{f_{ij}^* - f_{ij}^-} \right]^p \right) \right\}^{\frac{1}{p}} - \left(d_P^{NIS^B}\right)^-}{\left(d_P^{NIS^B}\right)^* - \left(d_P^{NIS^B}\right)^-} + D_B^{NIS^-} - D_B^{NIS^+} = 1 \tag{24}$$

$x \in \overline{S}(x)$

$\boldsymbol{x}_1 = \boldsymbol{x}_1^*$

6. The TOPSIS algorithm for FRBMNPP

The algorithm for the proposed two TOPSIS models for FRBMNPP follows as:

Step 1:	Ask the leader for an agreeable value of α.
Step 2:	Formulate the α-(FRBMNPP) for the UAM.
Step 3:	Calculate the individual minimum and maximum values for the objective functions.
Step 4:	Construct the PIS & NIS payoff tables of the leader problem Eq. (10).
Step 5:	Set up $d_2^{PIS^F}(x)$ and $d_2^{NIS^B}(x)$ according to Eq. (11).
Step 6:	Elicit the membership functions $\mu_{d_2^{PIS^F}}(x)$ and $\mu_{d_2^{NIS^F}}(x)$ Eqs. (14) and (15).
Step 7:	If the leader considers model (**I**) for solving the α-(FRBMNPP), then go to Step 8; otherwise he decide model (**II**), go to Step 9.
Step 8:	Formulate and solve the model (**I**), Eq. (16), for the leader α-(FRBMNPP) then go to Step 10.
Step 9:	Formulate and solve the model (**II**), Eq. (17), for the leader α-(FRBMNPP) then go to Step 10.
Step 10:	Set $\boldsymbol{x}_{1l} = \boldsymbol{x}_{1l}^* = \left(x_{11}^*, x_{12}^*, \cdots, x_{1n_1}^*\right)$.
Step 11:	Construct the PIS & NIS payoff tables of the α-(FRBMNPP) Eq. (6).
Step 12:	Set up $d_2^{PIS^B}(x)$ and $d_2^{NIS^B}(x)$ according to Eq. (18)&(19), respectively.
Step 13:	Elicit $\mu_{d_2^{PIS^B}}(x)$ and $\mu_{d_2^{NIS^B}}(x)$ Eq. (21) &(22).
Step 14:	Formulate and solve the model (**I**), Eq. (23), for the UAM of the α-(FRBMNPP), go to Step 16.
Step 15:	Formulate and solve the model (**II**), Eq. (24) for the UAM of the α-(FRBMNPP).
Step 16:	If the compromise solution belongs to $\underline{S}(\boldsymbol{x})$, then go to Step 17, else go to Step 18.
Step 17:	If the leader satisfied with the solution, then α-surely Pareto optimal solution obtained go to Step 20, else go to Step 19.
Step 18:	Solve the α-(FRBMNPP) for the LAM. If the leader satisfied with the solution, then α-possibly Pareto optimal solution obtained go to Step 20, else go to Step 19.
Step 19:	Modify the value of α, and go to Step 2.
Step 20:	End.

7. Numerical illustration

Consider the following FRBMNPP where fuzziness in the objective functions and the set of constraint modeled as a rough environment:

[**Leader**]

$$\underbrace{\max}_{\boldsymbol{x}_1}\begin{pmatrix}\tilde{f}_{11}(x) = \tilde{c}_1x_1^2 + \tilde{c}_2x_2^3 + \tilde{c}_3,\\ \tilde{f}_{12}(x) = \tilde{c}_3x_1^2 + \tilde{c}_4x_1x_2\end{pmatrix},$$

where x_2, solves

[**Follower**]

$$\underbrace{\max}_{\boldsymbol{x}_2}\begin{pmatrix}\tilde{f}_{21}(x) = \tilde{c}_4x_1^2 + \tilde{c}_1x_1x_2^2,\\ \tilde{f}_{22}(x) = \tilde{c}_1x_1^3 + \tilde{c}_3x_2^2\end{pmatrix},$$

subject to

$x \in S(x)$

$$\left\{\begin{matrix} x_1^2 + x_2^2 \le 16\\ x_1 + x_2 \le 5\\ x_1, x_2 \ge 0\end{matrix}\right\} \subseteq S(x) \subseteq \left\{\begin{matrix} x_1^2 + x_2^2 \le 36\\ x_1 \le 5.5,\\ x_2 \le 5.5\\ x_1, x_2 \ge 0\end{matrix}\right\}$$

$$\mu_{\tilde{c}_1} = \begin{cases} 0 & x < 5,\\ (x^2 - 25)/11 & 5 \le x < 6,\\ 1 & x = 6,\\ (64 - x^2)/28 & 6 < x \le 8,\\ 0 & x > 8,\end{cases}$$

$$\mu_{\tilde{c}_2} = \begin{cases} 0 & x < 2,\\ (x^2 - 4)/5 & 2 \le x < 3,\\ 1 & x = 3,\\ (25 - x^2)/16 & 3 < x \le 5,\\ 0 & x > 5,\end{cases}$$

$$\mu_{\tilde{c}_3} = \begin{cases} 0 & x < 3,\\ (x^2 - 9)/7 & 3 \le x < 4,\\ 1 & x = 4,\\ (36 - x^2)/20 & 4 < x \le 6,\\ 0 & x > 6,\end{cases}$$

$$\mu_{\tilde{c}_4} = \begin{cases} 0 & x < 0.5,\\ (x^2 - 0.25)/0.75 & 0.5 \le x < 1,\\ 1 & x = 1,\\ (4 - x^2)/3 & 1 < x \le 2,\\ 0 & x > 2,\end{cases}$$

For an agreeable value of let $\alpha = 0.2$ then the FRBMNPP would be transferred into the following α-(FRBMNPP):

[**Leader**]

$$\underbrace{\max}_{\boldsymbol{x}_1}\begin{pmatrix}\left(\tilde{f}_{11}\right)_\alpha^U(x) = \sqrt{64 - 28\alpha}x_1^2 + \sqrt{25 - 16\alpha}x_2^3 + \sqrt{36 - 20\alpha},\\ \left(\tilde{f}_{12}\right)_\alpha^U(x) = \sqrt{36 - 20\alpha}x_1^2 + \sqrt{4 - 3\alpha}x_1x_2\end{pmatrix},$$

where x_2, solves

[**Follower**]

$$\underbrace{\boldsymbol{x}_2 \max}\left(\begin{array}{c}\left(\tilde{f}_{21}\right)^U_\alpha(x) = \sqrt{4-3\alpha}x_1^2 + \sqrt{25-16\alpha}x_1x_2^2,\\ \left(\tilde{f}_{22}\right)^U_\alpha(x) = \sqrt{64-28\alpha}x_1^3 + \sqrt{36-20\alpha}x_2^2\end{array}\right),$$

subject to

$x \in S(x)$

$$\left\{\begin{array}{c}x_1^2+x_2^2 \le 16\\ x_1+x_2 \le 5\\ x_1,x_2 \ge 0\end{array}\right\} \subseteq S(\boldsymbol{x}) \subseteq \left\{\begin{array}{c}x_1^2+x_2^2 \le 36\\ x_1 \le 5.5,\\ x_2 \le 5.5\\ x_1,x_2 \ge 0\end{array}\right\}$$

We solve the FRBMNPP using the KKT method. First, the UAM of the α-(FRBMNPP), the KKT condition applied to obtain the single level problem as:

$$max w_1\left(7.64x_1^2+4.67x_2^3+5.66\right)+(1-w_1)(5.66x_1^2+1.84x_1x_2)$$

subject to

$$w_2(9.34x_1x_2)+(1-w_2)(11.31x_2)+2\mu_1x_2+\mu_2-\mu_4=0$$

$$\mu_1\left(x_1^2+x_2^2-36\right)=0$$

$$\mu_2(x_2-5.5)=0$$

$$\mu_3(x_1)=0$$

$$\mu_4(x_2)=0$$

$$x_1^2+x_2^2 \le 36$$

$$x_1 \le 5.5$$

$$x_2 \le 5.5$$

$$x_1 \ge 0$$

$$x_2 \ge 0$$

$$\mu_1,\mu_2,\mu_3,\mu_4 \ge 0,\quad w_1 \in [0,1],\quad w_2 \in [0,1]$$

Using Lingo 18, the solution of the KKT method for the UAM of theα-(FRBMNPP), is $x^* = (0, 5.5)$, the objective values for different values of Alpha are given in Table 6.

Solution using the two models of TOPSIS approach. The individual maximum and minimum values are summarized in Table 1. The PIS & NIS payoff tables for the leader problem are given in Tables 2 and 3 respectively.

Assume that $\gamma_{11}=\gamma_{12}=0.5$, then the equation of $d_p^{PIS^F}(x)$ and $d_P^{NIS^F}(x)$ when $p=2$ are:

$$d_2^{PIS^F}(x)=\left\{\begin{array}{c}(0.5)^2\left[\frac{1013.74-\left(7.64x_1^2+4.67x_2^3+5.66\right)}{1013.74-5.66}\right]^2\\ +(0.5)^2\left[\frac{195.48-\left(5.66x_1^2+1.84x_1x_2\right)}{195.48-0}\right]^2\end{array}\right\}^{\frac{1}{2}}$$

$$d_2^{NIS^F}(x)=\left\{\begin{array}{c}(0.5)^2\left[\frac{\left(7.64x_1^2+4.67x_2^3+5.66\right)-5.66}{1013.74-5.66}\right]^2\\ +(0.5)^2\left[\frac{\left(5.66x_1^2+1.84x_1x_2\right)-0}{195.48-0}\right]^2\end{array}\right\}^{\frac{1}{2}}$$

$$d_2^{PIS^F}(x)=\Big\{2.46*10^{-7}\left[1013.74-(f_{11}(\boldsymbol{x}))^U_{0.2}\right]^2 +6.54*10^{-6}\left[195.48-(f_{12}(\boldsymbol{x}))^U_{0.2}\right]^2\Big\}^{\frac{1}{2}}$$

$$d_2^{NIS^F}(x)=\Big\{2.46*10^{-7}\left[(f_{11}(\boldsymbol{x}))^U_{0.2}-5.66\right]^2 +6.54*10^{-6}\left[(f_{12}(\boldsymbol{x}))^U_{0.2}-0\right]^2\Big\}$$

Thus we compute $\max d_2^{PIS^F}(x)=0.7070$, $\min d_2^{PIS^F}(x)=0.296$, and $\max d_2^{NIS^F}(x)=0.521$, $\min d_2^{NIS^F}(x)=0$. Then we have $d_2^{F^*}=(0.296,0.521)$ and $d_2^{F^-}=(0.7070,0)$, therefore, $\mu_{d_2^{PIS^F}}(x)$ *and* $\mu_{d_2^{NIS^F}}(x)$ can be obtained as:

$$\mu_{d_2^{PIS^F}}(x)=1.72-2.43*d_2^{PIS^F}(x)$$

$$\mu_{d_2^{NIS^F}}(x)=1.92*d_2^{NIS^F}(x)$$

The proposed two TOPSIS models of the leader UAM:

Model (I) Fuzzy max–min method	**Model (II) FGP method**
$\max \lambda$	$\min Z = D_F^{PIS^+}+D_F^{NIS^-}$
subject to	subject to
$1.72-2.43*d_2^{PIS^F}(x) \ge \lambda$	$1.72-2.43*d_2^{PIS^F}(x)+D_F^{PIS^-}-D_F^{PIS^+}=1$
$1.92*d_2^{NIS^F}(x) \ge \lambda$	$1.92*d_2^{NIS^F}(x)+D_F^{NIS^-}-D_F^{NIS^+}=1$
$x_1^2+x_2^2 \le 36$	$x_1^2+x_2^2 \le 36$
$x_1 \le 5.5$	$x_1 \le 5.5$
$x_2 \le 5.5$	$x_2 \le 5.5$
$x_1,x_2 \ge 0,\quad \lambda \in [0,1]$	$x_1,x_2 \ge 0,\quad D_F^{PIS^-},D_F^{PIS^+},D_F^{NIS^-},D_F^{NIS^+} \ge 0,$
$\lambda = 0.92,\quad x^* = (5.051, 3.239).$	$x^* = (5.4999, 2.398).$

Table 1 Individual maximum, minimum values.

	$(f_{11}(\mathbf{x}))_{0.2}^{U}$	$(f_{12}(\mathbf{x}))_{0.2}^{U}$	$(f_{21}(\mathbf{x}))_{0.2}^{U}$	$(f_{22}(\mathbf{x}))_{0.2}^{U}$
$\max(f_{kj}(\mathbf{x}))_{0.2}^{U}$	1013.74	195.48	832.63	1442.32
$\min(f_{kj}(\mathbf{x}))_{0.2}^{U}$	5.66	0	0	0

Table 2 PIS payoff table of the leader.

	$(f_{11}(\mathbf{x}))_{0.2}^{U}$	$(f_{12}(\mathbf{x}))_{0.2}^{U}$	x_1	x_2
$\max\,(f_{11}(\mathbf{x}))_{0.2}^{U}$	1013.74^*	–	5.5	5.5
$\max\,(f_{12}(\mathbf{x}))_{0.2}^{U}$	301.17	195.48^*	5.5	2.398

$F^* = (f_{11}^*, f_{12}^*) = (1013.74; 195.48).$

Table 3 NIS payoff table of the leader.

	$(f_{11}(\mathbf{x}))_{0.2}^{U}$	$(f_{12}(\mathbf{x}))_{0.2}^{U}$	x_1	x_2
$\min(f_{11}(\mathbf{x}))_{0.2}^{U}$	5.66^*	0	0	0
$\min(f_{12}(\mathbf{x}))_{0.2}^{U}$	11.88	0^*	0	1.1

$F^- = (f_{11}^-, f_{12}^-) = (5.66, 0).$

Using Lingo 18, the compromise solution of the leader UAM is $x^* = (5.051, 3.239)$. for model (I) and $x^* = (5.4999, 2.398)$ for model (II). Tables 4 and 5 show the PIS and NIS payoff tables for the follower problem.

Assume that $\gamma_{kj} = 0.25$, then the equation of $d_p^{PIS^B}(x)$ and $d_P^{NIS^B}(x)$ when $p = 2$ are:

Table 4 PIS payoff table of the follower.

	$(f_{21}(\mathbf{x}))_{0.2}^{U}$	$(f_{22}(\mathbf{x}))_{0.2}^{U}$	x_1	x_2
$\max(f_{21}(\mathbf{x}))_{0.2}^{U}$	832.63^*	1442.32	5.5	5.5
$\max(f_{22}(\mathbf{x}))_{0.2}^{U}$	832.63	1442.32^*	5.5	5.5

$F^* = (f_{21}^*, f_{22}^*) = (832.63; 1442.32).$

Table 5 NIS payoff table of the follower.

	$(f_{21}(\mathbf{x}))_{0.2}^{U}$	$(f_{22}(\mathbf{x}))_{0.2}^{U}$	x_1	x_2
$\min(f_{21}(\mathbf{x}))_{0.2}^{U}$	0^*	0	0	0
$\min(f_{22}(\mathbf{x}))_{0.2}^{U}$	0	0^*	0	0

$F^- = (f_{21}^-, f_{22}^-) = (0, 0).$

$$d_2^{PIS^B}(x) = \left\{ \begin{array}{l} (0.25)^2 \left[\frac{1013.74 - (7.64x_1^2 + 4.67x_2^3 + 5.66)}{1013.74 - 5.66} \right]^2 \\ +(0.25)^2 \left[\frac{195.48 - (5.66x_1^2 + 1.84x_1x_2)}{195.48 - 0} \right]^2 \\ +(0.25)^2 \left[\frac{832.63 - (1.84x_1^2 + 4.67x_1x_2^2)}{832.63 - 0} \right]^2 \\ +(0.25)^2 \left[\frac{1442.32 - (7.64x_1^3 + 5.66x_2^2)}{1442.32 - 0} \right]^2 \end{array} \right\}^{\frac{1}{2}}$$

$$d_2^{NIS^B}(x) = \left\{ \begin{array}{l} (0.25)^2 \left[\frac{(7.64x_1^2 + 4.67x_2^3 + 5.66) - 5.66}{1013.74 - 5.66} \right]^2 \\ +(0.25)^2 \left[\frac{(5.66x_1^2 + 1.84x_1x_2) - 0}{195.48 - 0} \right]^2 \\ +(0.25)^2 \left[\frac{(1.84x_1^2 + 4.67x_1x_2^2) - 0}{832.63 - 0} \right]^2 \\ +(0.25)^2 \left[\frac{(7.64x_1^3 + 5.66x_2^2) - 0}{1442.32 - 0} \right]^2 \end{array} \right\}^{\frac{1}{2}}$$

$$d_2^{PIS^B}(x) = \left\{ \begin{array}{l} 6.15 * 10^{-8} \left[1013.74 - (f_{11}(\mathbf{x}))_{0.2}^{U} \right]^2 \\ +1.64 * 10^{-6} \left[195.48 - (f_{12}(\mathbf{x}))_{0.2}^{U} \right]^2 \\ +9 * 10^{-8} \left[832.63 - (f_{21}(\mathbf{x}))_{0.2}^{U} \right]^2 \\ +3 * 10^{-8} \left[1442.32 - (f_{22}(\mathbf{x}))_{0.2}^{U} \right]^2 \end{array} \right\}^{\frac{1}{2}}$$

$$d_2^{NIS^B}(x) = \left\{ \begin{array}{l} 6.15 * 10^{-8} \left[(f_{11}(\mathbf{x}))_{0.2}^{U} - 5.66 \right]^2 \\ +1.64 * 10^{-6} \left[(f_{12}(\mathbf{x}))_{0.2}^{U} - 0 \right]^2 \\ +9 * 10^{-8} \left[(f_{21}(\mathbf{x}))_{0.2}^{U} - 0 \right]^2 \\ +3 * 10^{-8} \left[(f_{22}(\mathbf{x}))_{0.2}^{U} - 0 \right]^2 \end{array} \right\}^{\frac{1}{2}}$$

Table 6 Comparison between the FGP method, Fuzzy max- min method and KKT method for the UAM.

α	FGP method	Fuzzy max-min method	KKT method
$\alpha = 0$	$f_{11} = 316.93$ $f_{12} = 207.86$ $f_{21} = 218.62$ $f_{22} = 1365.36$ $(x_1, x_2) = (5.4998, 2.397)$	$f_{11} = 382.91$ $f_{12} = 184.8$ $f_{21} = 319.18$ $f_{22} = 1083.36$ $(x_1, x_2) = (5.032, 3.267)$	$f_{11} = 837.875$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 181.5$ $(x_1, x_2) = (0, 5.5)$
$\alpha = 0.2$	$f_{11} = 298.77$ $f_{12} = 195.17$ $f_{21} = 199.68$ $f_{22} = 1302.77$ $(x_1, x_2) = (5.4999, 2.368)$	$f_{11} = 359.11$ $f_{12} = 174.49$ $f_{21} = 294.25$ $f_{22} = 1043.86$ $(x_1, x_2) = (5.05, 3.238)$	$f_{11} = 782.47$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 171.12$ $(x_1, x_2) = (0, 5.5)$
$\alpha = 0.4$	$f_{11} = 285.1$ $f_{12} = 181.73$ $f_{21} = 188$ $f_{22} = 1235.68$ $(x_1, x_2) = (5.493, 2.411)$	$f_{11} = 337.54$ $f_{12} = 162.14$ $f_{21} = 271.37$ $f_{22} = 990.8$ $(x_1, x_2) = (5.048, 3.243)$	$f_{11} = 722.829$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 160.068$ $(x_1, x_2) = (0, 5.5)$
$\alpha = 0.5$	$f_{11} = 730.57$ $f_{12} = 49.53$ $f_{21} = 305.43$ $f_{22} = 249.46$ $(x_1, x_2) = (2.379, 5.5)$	$f_{11} = 731.18$ $f_{12} = 50.13$ $f_{21} = 307.81$ $f_{22} = 251.64$ $(x_1, x_2) = (2.397, 5.5)$	$f_{11} = 690.57$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 154.28$ $(x_1, x_2) = (0, 5.5)$
$\alpha = 0.6$	$f_{11} = 422.95$ $f_{12} = 116.18$ $f_{21} = 324.1$ $f_{22} = 623.58$ $(x_1, x_2) = (4.275, 4.21)$	$f_{11} = 307.28$ $f_{12} = 151.3$ $f_{21} = 238.12$ $f_{22} = 960.24$ $(x_1, x_2) = (5.1, 3.16)$	$f_{11} = 657.802$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 148.194$ $(x_1, x_2) = (0, 5.5)$
$\alpha = 0.8$	$f_{11} = 247.58$ $f_{12} = 151.82$ $f_{21} = 148.31$ $f_{22} = 1098.78$ $(x_1, x_2) = (5.5, 2.396)$	$f_{11} = 276.72$ $f_{12} = 138.76$ $f_{21} = 202.87$ $f_{22} = 926.16$ $(x_1, x_2) = (5.156, 3.068)$	$f_{11} = 585.595$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 135.282$ $(x_1, x_2) = (0, 5.5)$
$\alpha = 1$	$f_{11} = 226.8$ $f_{12} = 134.18$ $f_{21} = 125.05$ $f_{22} = 1021.23$ $(x_1, x_2) = (5.5, 2.397)$	$f_{11} = 245.26$ $f_{12} = 124.32$ $f_{21} = 164.53$ $f_{22} = 887.04$ $(x_1, x_2) = (5.217, 2.962)$	$f_{11} = 503.125$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 121$ $(x_1, x_2) = (0, 5.5)$

So, $\max d_2^{PIS^B}(x) = 0.4999$, $\min d_2^{PIS^B}(x) = 0.235$, and $\max d_2^{NIS^B}(x) = 0.35$, $\min d_2^{NIS^B}(x) = 0$. Thus we have $d_2^{B^*} = (0.235, 0.35)$ and $d_2^{B^-} = (0.4999, 0)$, so, $\mu_{d_2^{PIS^B}}(x)$ and $\mu_{d_2^{NIS^B}}(x)$ can be obtained as:

$$\mu_{d_2^{PIS^B}}(x) = 1.88 - 3.775 * d_2^{PIS^B}(x)$$

$$\mu_{d_2^{NIS^B}}(x) = 2.86 * d_2^{NIS^B}(x)$$

The proposed two TOPSIS models for FRBMNPP:

Model (I) Fuzzy max–min method	**Model (II)** FGP **method**
$\max \delta$ subject to $1.72 - 2.43 * d_2^{PIS^B}(x) \geq \delta$ $1.92 * d_2^{NIS^B}(x) \geq \delta$ $x_1^2 + x_2^2 \leq 36$ $x_1 = 5.051$ $x_2 \leq 5.5$ $x_1, x_2 \geq 0, \delta \in [0, 1]$	$\min Z = D_B^{PIS^+} + D_B^{NIS^-}$ subject to $1.88 - 3.775 * d_2^{PIS^B}(x) + D_B^{PIS^-} - D_B^{PIS^+} = 1$ $2.86 * d_2^{NIS^B}(x) + D_B^{NIS^-} - D_B^{NIS^+} = 1$ $x_1^2 + x_2^2 \leq 36$ $x_1 = 5.4999$ $x_2 \leq 5.5$ $D_B^{PIS^-}, D_B^{PIS^+}, D_B^{NIS^-}, D_B^{NIS^+} \geq 0,$
$\delta = 0.896$, $x^* = (5.051, 3.238)$.	$x^* = (5.4999, 2.368)$.

Table 7 Comparison between the FGP method, Fuzzy max- min method and KKT method for the LAM.

α	FGP method	Fuzzy max-min method	KKT method
$\alpha = 0$	$f_{11} = 132.16$ $f_{12} = 98.56$ $f_{21} = 38.42$ $f_{22} = 496.7$ $(x_1, x_2) = (3.954, 0.6016)$	$f_{11} = 326$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 96$ $(x_1, x_2) = (0, 4)$	$f_{11} = 326$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 96$ $(x_1, x_2) = (0, 4)$
$\alpha = 0.2$	$f_{11} = 126.33$ $f_{12} = 92.67$ $f_{21} = 33.77$ $f_{22} = 477.36$ $(x_1, x_2) = (3.964, 0.5125)$	$f_{11} = 126.97$ $f_{12} = 92.57$ $f_{21} = 32.1$ $f_{22} = 482.56$ $(x_1, x_2) = (3.98, 0.399)$	$f_{11} = 304.476$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 90.51$ $(x_1, x_2) = (0, 4)$
$\alpha = 0.4$	$f_{11} = 120.1$ $f_{12} = 86.68$ $f_{21} = 31.61$ $f_{22} = 453.14$ $(x_1, x_2) = (3.96, 0.564)$	$f_{11} = 120.63$ $f_{12} = 86.13$ $f_{21} = 28.57$ $f_{22} = 458.99$ $(x_1, x_2) = (3.98, 0.3518)$	$f_{11} = 281.309$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 84.664$ $(x_1, x_2) = (0, 4)$
$\alpha = 0.5$	$f_{11} = 118.16$ $f_{12} = 81.86$ $f_{21} = 25.3$ $f_{22} = 452.15$ $(x_1, x_2) = (3.999, 0.0489)$	$f_{11} = 117.53$ $f_{12} = 83.13$ $f_{21} = 26.99$ $f_{22} = 448$ $(x_1, x_2) = (3.985, 0.3405)$	$f_{11} = 268.78$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 81.6$ $(x_1, x_2) = (0, 4)$
$\alpha = 0.6$	$f_{11} = 113.45$ $f_{12} = 80.13$ $f_{21} = 27.7$ $f_{22} = 429.31$ $(x_1, x_2) = (3.964, 0.535)$	$f_{11} = 114.19$ $f_{12} = 79.82$ $f_{21} = 25.25$ $f_{22} = 435.62$ $(x_1, x_2) = (3.986, 0.334)$	$f_{11} = 256.053$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 78.384$ $(x_1, x_2) = (0, 4)$
$\alpha = 0.8$	$f_{11} = 106.82$ $f_{12} = 72.82$ $f_{21} = 22.27$ $f_{22} = 407.08$ $(x_1, x_2) = (3.979, 0.4093)$	$f_{11} = 107.62$ $f_{12} = 71.93$ $f_{21} = 20.26$ $f_{22} = 412.52$ $(x_1, x_2) = (3.999, 0.0894)$	$f_{11} = 228.015$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 71.554$ $(x_1, x_2) = (0, 4)$
$\alpha = 1$	$f_{11} = 99.12$ $f_{12} = 64.93$ $f_{21} = 17.91$ $f_{22} = 378.11$ $(x_1, x_2) = (3.977, 0.4198)$	$f_{11} = 99.37$ $f_{12} = 64.91$ $f_{21} = 17.39$ $f_{22} = 379.92$ $(x_1, x_2) = (3.984, 0.3572)$	$f_{11} = 196$ $f_{12} = 0$ $f_{21} = 0$ $f_{22} = 64$ $(x_1, x_2) = (0, 4)$

Using Lingo 18, the compromise solution of the α-(FRBMNPP) UAM is $x^* = (5.051, 3.238)$ for model (I) and $x^* = (5.4999, 2.368)$ for model (II). The objective values for different Alpha are given in Table 6.

Since the obtained solution of the KKT method and the two models of TOPSIS approach for the UAM $x^* \notin \underline{S}(\boldsymbol{x})$. So we have to solve the LAM by using the previous methods for different values of α thus the solution obtained is an α-possibly feasible and an α-possibly Pareto optimal solution. The results for the UAM and LAM were listed in Tables 6 and 7 respectively.

The results of the proposed FGP method, fuzzy max-min method and the KKT method, in Tables 6 and 7, for solving the UAM and LAM, respectively of the FRBMNPP indicate that the FGP method and fuzzy max-min method are close to one another and preferred than the KKT method.

8. Discussion

The α-level and Roughness

1. The α-level: A specific α-level is adopted in the proposed methods to represent the confidence level on DMs' subjective uncertainty to specify parameter values in the FRBMNPP. For simplification, the α-level for all parameters in the solution process are considered to be the same. However, these may be limitations in practical applications. The determination of α-levels for various DMs' subjective uncertainties could be different in the real world, due to DMs' different consideration of the corresponding problems. Thus, we solve the UAM and LAM of the α-(FRBMNPP) for different α-levels.

2. Roughness: in our proposed FRBMNPP we deal with uncertainty by giving two boundary regions for the set of constrains based on RST. Thus after formulating the α-(FRBMNPP) we transform the problem into an UAM and LAM and solve for different values of α.

9. Conclusion

In this study, we architect FRBMNPP, moreover a strategy for tackling such problem is suggested. Firstly, we formulate theα-(FRBMNPP) then we obtain the UAM and LAM. The KKT method and two models of TOPSIS approach based on Fuzzy max-min method and FGP method were applied to solve this problem. We suggest a strategy to solve such problem by firstly solve the UAM is its solution belongs to the lower approximation set then the solution is reached otherwise we solve the LAM. We apply this strategy for different values of α to ensure the validation of the proposed methods and algorithm. A numerical illustration was presented to clarify the efficiency of the proposed methods.

Several open points for research in the area of multi-level optimization, from our point of view, to be studied in the future. Some of these points are given in the following:

1. Fuzzy rough fractional ML-DMP should be put on spot via interactive algorithms.
2. Practical fuzzy rough BL-DMP is a vital field in the future researches.

Declaration of Competing Interest

The authors declare that they have no competing interests.

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